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A Secant Based Roots Finding Algorithm Design and its Applications to Circular Waveguides

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Abstract: *Determination of zeros of first kind of Bessel functions and their derivatives by fast and reliable calculation techniques is essential to determine the necessary Transverse Electric (TE) and Transverse Magnetic(TM) modes supported by the circular waveguide (CWG). Here, a fast and reliable computational algorithm design based on the conventional secant method to determine the first n roots of these special functions is being presented. Our design is based on scanning the guide radius with the desired number of sub-intervals and finding their zerosby the secant method to determine the desired mode, TE or TM. Repeated roots and roots out of the domain is rejected and the remaining desired roots are used to determine the cut-off frequency and propagating guide frequency in the z direction. Our design runs under the free "Wolfram CDF (Computable Document Format) player" software where the circular waveguide parameters and other designed parameters for our secant based calculation can be selected by the user via the manipulation console. It also has been opened to the useof researchers for free in the web page of our institution as being presented here.*

Keywords: *Bessel functions, Circular waveguides, Cylindrical waveguides, TE modes, TM modes, Numerical root finding, Secant method.*

I. Introduction

Determination of Transverse Electric (TE) and Transverse Magnetic (TM) modes supported by the circular waveguides (CWGs) involve finding roots of Bessel functions of first kind and their derivatives [1-6]. However, since these functions are conventionally and analytically defined in the form of infinite series, finding their zeros involve various techniques for fast and reliable approximations [7-13]. Here we design an algorithm by using the numerical secant method, which is among such numerical approximation methods [14-16], to find all possible modes for a specific circular waveguide whose parameters can be selected by the user. A general appearance of our design running under the free Mathematicabased Wolfram CDF player (Computable Document Format) software is given in Fig. 1. It is freely open to the use of researchers from all around the worldby downloading it from the web address of our institution given in [17]. General properties of the Wolfram CDF player is available in [18-21]Since the secant method is usedto find a single zero of a function for a given intervalin the domain (in other words, a pair of points in the domain), our suggestion involves scanning the domain (guide radius) according tothe user selected number of divisionsto determine the necessary intervals (or, point pairs) to apply the conventional secant algorithm simultaneously to these intervals to determine the related modes in each divisions. These modes are obtained in the ascending order as desired. Moreover,related cut-off frequencies and propagating wave frequencies (in the z direction) are also calculated directly in the ascending order according to the waveguide parameters in our work being presented here.

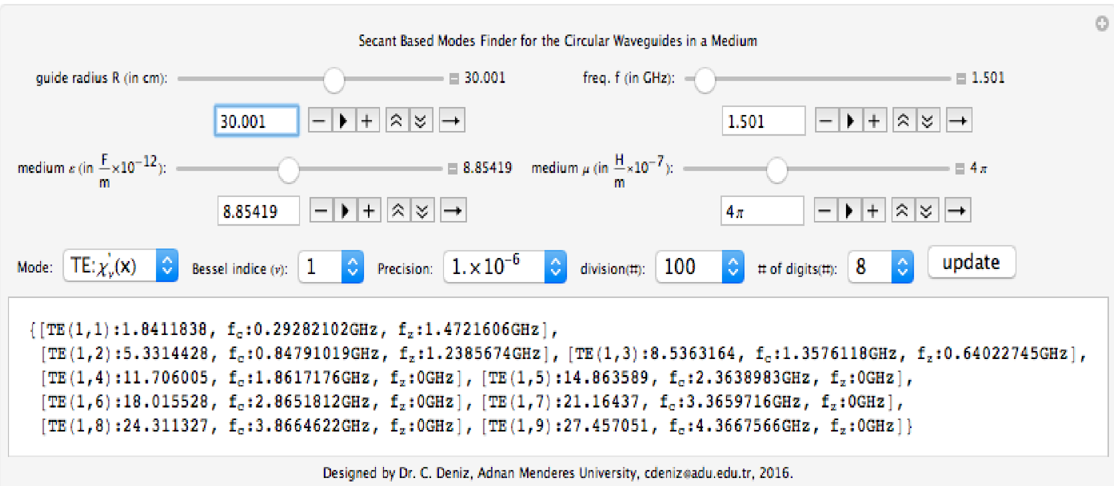

Fig. 1. An appearance of the designed software while running

II. Circular Waveguides

The circular waveguide is occasionally used as an alternative to the rectangular waveguide. Like other waveguides constructed from a single, enclosed conductor, the circular waveguide supports Transverse Electric (TE) and Transverse Magnetic (TM) modes. Note that, Transverse Electromagnetic(TEM) mode supported by other kinds of waveguides such as coaxial, parallel plate, etc., is not supported in the circular waveguide [1-6]. These two kinds of modes for the CWG are determined from the Bessel functions of integer orders and their derivatives, whose graphs for first five index values are given in Fig. 2.

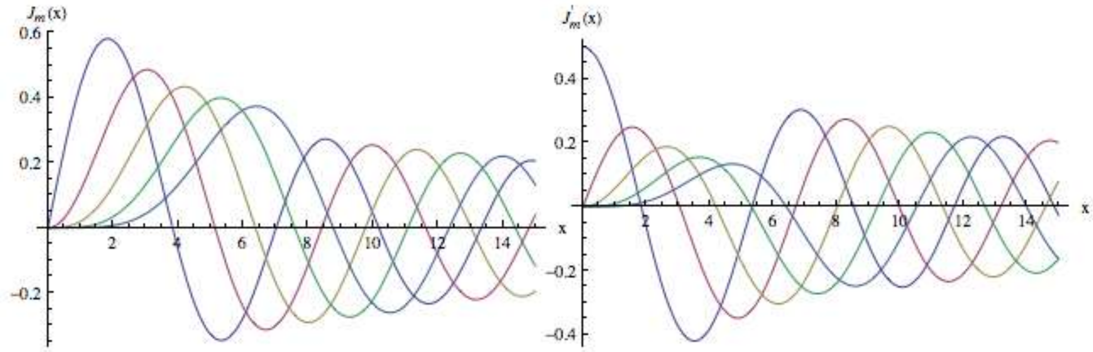

Fig.2. Graphs of Bessel functions of first type: $J_m(x)$, with index values $m = 0,1,2,3,4,5$ and their derivatives:$J'_m(x)$.

These modes have characteristic cut-off frequencies, below which electromagnetic energy is severely attenuated. General properties of the TE and TM modes can be summarized as follows [1-6]:

2.1 TE Modes

The Transverse Electric to z (TE^z) modes(or simply and conventionally TE modes) can be derived by letting the vector potential $\boldsymbol{A}$ and $\boldsymbol{F}$ be equal to the followings:

$$\boldsymbol{A} = 0,\ \ \boldsymbol{F} = \boldsymbol{a_z} F_z(\rho, \emptyset, z) \quad (1)$$

By using the following boundary conditions:

$$E_\emptyset(\rho = R, \emptyset, z) = 0,\ E_z(\rho = R, \emptyset, z) = 0 \quad (2)$$

We finally obtain β_z of the mn mode as follows:

$$TE \Rightarrow \beta_\rho^2 + \beta_z^2 = \beta^2 \quad (3a)$$

$$TE \Rightarrow (\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{\beta^2 - \left(\frac{\chi'_{mn}}{R}\right)^2}, \beta > \beta_\rho = \frac{\chi'_{mn}}{R} \\ 0, \qquad \beta = \beta_c = \beta_\rho = \frac{\chi'_{mn}}{R} \\ -j\sqrt{\beta_\rho^2 - \beta^2} = j\sqrt{\left(\frac{\chi'_{mn}}{R}\right)^2 - \beta^2}, \beta < \beta_\rho = \frac{\chi'_{mn}}{R} \end{cases} \quad (3b)$$

where χ'_{mn} represents the nth zero ($n = 1, 2, 3, \dots$) of the derivative of the Bessel function $J_m(x)$ of the first kind and of order m ($m = 0, 1, 2, 3, \dots$). The smallest value of χ'_{mn} corresponds to $\chi'_{11} = 1.8412$ (where $m = 1, n = 1$). Cutoff is defined when $\beta_{z(mn)} = 0$, namely:

$$TE \Rightarrow \beta_c = \omega_c\sqrt{\mu\varepsilon} \Rightarrow (f_c)_{mn} = \frac{\chi'_{mn}}{2\pi R\sqrt{\mu\varepsilon}} \quad (3c)$$

where $(f_c)_{mn}$ is the cut-off frequency above which the related TE mode propogates with the guide wavelength:

$$TE \Rightarrow \lambda_g = \frac{2\pi}{(\beta_z)_{mn}} \quad (3d)$$

2.2 TM Modes

Similarly, the Transverse Magnetic to z (TM^z) modes (or simply and conventionally TM modes) can be derived by letting the vector potential $\boldsymbol{A}$ and $\boldsymbol{F}$ be equal to the followings:

$$\boldsymbol{F} = 0,\ \ \boldsymbol{A} = \boldsymbol{a_z} A_z(\rho, \emptyset, z) \quad (4)$$

By using the following boundary conditions:

$$E_\emptyset(\rho = R, \emptyset, z) = 0,\ E_z(\rho = R, \emptyset, z) = 0 \quad (5)$$

we finally obtain β_z of the mn mode as follows:

$$TM \Rightarrow \beta_\rho^2 + \beta_z^2 = \beta^2 \quad (6a)$$

$$TM \Rightarrow (\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2-\beta_\rho^2} = \sqrt{\beta^2-\left(\frac{\chi_{mn}}{R}\right)^2}, \beta > \beta_\rho = \frac{\chi_{mn}}{R} \\ 0, \quad \beta = \beta_c = \beta_\rho = \frac{\chi_{mn}}{R} \\ -j\sqrt{\beta_\rho^2-\beta^2} = j\sqrt{\left(\frac{\chi_{mn}}{R}\right)^2-\beta^2}, \beta < \beta_\rho = \frac{\chi_{mn}}{R} \end{cases} \tag{6b}$$

where χ_{mn} represents the nth zero ($n = 1, 2, 3, ...$) of the Bessel function $J_m(x)$ of the first kind and of order m ($m = 0, 1, 2, 3, ...$). The smallest value of χ_{mn} corresponds to $\chi_{01} = 2.4049$ (where $m = 0, n = 1$). Cut-off is defined when $\beta_{z(mn)} = 0$, namely:

$$TM \Rightarrow \beta_c = \omega_c\sqrt{\mu\varepsilon} \Rightarrow (f_c)_{mn} = \frac{\chi_{mn}}{2\pi R\sqrt{\mu\varepsilon}} \tag{6c}$$

where $(f_c)_{mn}$ is the cut-off frequency above which the related TM mode propogates with the guide wavelength given in (6c). Since the cutoff frequencies of the TE_{0n} and TM_{1n} modes are identical ($\chi'_{0n} = \chi_{1n}$); they are referred to also as degenerate modes. Similar to the TE modes, guide wavelength is as follows:

$$TM \Rightarrow \lambda_g = \frac{2\pi}{(\beta_z)_{mn}} \tag{6d}$$

III. Our Secant Based Design For Finding Modes In Circular Waveguides

3.1. The Secant Method For Finding a Single Root

The secant method is an algorithm used to approximate the root of a given function in the interval: $[x_1, x_2]$ by iterations. The method is based on approximating the function by using secant lines. Its algorithm is given in any fundamental textbooks such as [14-16]. To find a zero of a function by the secant method is today available in any computational program with a very simple commands such as the following for Mathematica [21-25]:

$$FindRoot[f, \{\rho, \rho_1, \rho_2, \boldsymbol{\rho_{min}}, \boldsymbol{\rho_{max}}\}, \boldsymbol{Method} \rightarrow \text{"}\boldsymbol{Secant}\text{"}, \boldsymbol{WorkingPrecision} -> p, AccuracyGoal \rightarrow g] \tag{7}$$

Command in (7) finds the root of function f by using initial ρ values ρ_1&ρ_2(as initial point sets to start the iteration) with the following optional terms: i) ρ_{min}, ρ_{max} values tosearch the roots in this interval only,ii) Methodis already secant as default sinceρ_1, ρ_2 both exists, iii) working precision and accuracygoal values $p\&g$. Optional terms are shown in bold in (7).The method would be defined as Newton-Raphson when ρ_2 is not entered as default [21,25]. Default values of working precision and accuracy goal as explained in [21,25] are also used when these options are not entered.

Approximation to the numerical value to π via root finding by the secant method is given in the following examples to summarize the options we use in our design by Mathematica:

3.1.1. Example 1:

In[1]=: FindRoot[Sin[ρ], {ρ, 1, 4}, Method -> "Secant", AccuracyGoal -> 6, WorkingPrecision -> 4]

$Out[1] = \{\rho -> 3.140\}$ (8a)

3.1.2. Example 2:

In[2]=: FindRoot[Sin[ρ], {ρ, 1, 4}, Method -> "Secant", AccuracyGoal -> 6, WorkingPrecision -> 18]

$Out[2] = \{\rho -> 3.14159265358975762\}$ (8b)

3.1.3. Example 3:

In[3]=: FindRoot[Sin[ρ], {ρ, 1, 4}, Method -> "Secant", AccuracyGoal -> 16, WorkingPrecision -> 18]

$Out[3] = \{\rho -> 3.14159265358979324\}$ (8c)

3.1.4. Example 4:

In[4]=: FindRoot[Sin[ρ], {ρ, 1, 4}, Method -> "Secant", AccuracyGoal -> 30, WorkingPrecision ->30]

$Out[4] = \{\rho -> 3.14159265358979323846264338328\}$ (8d)

3.1.5. Example 5:

$$In[5] =: FindRoot[Sin[\rho], \{\rho, 1, 3, 4, 5\}]$$

$Out[5] = FindRoot[Sin[\rho], \{\rho, 1, 3, 4, 5\}]$ (8e)

This ends up with the following error message:

$$During evaluation of In[5] :=$$

$FindRoot::streg$: "$The\ starting\ point\ \{1.\}\ is\ not\ in\ the\ search\ region\ \{\{4.\}, \{5.\}\}. >>$.(8f)

3.1.6. Example 6:

$$In[6] =: FindRoot[Sin[\rho], \{\rho, 1, 3, 3, 5\}]$$

$Out[6] = FindRoot[Sin[\rho], \{\rho, 1, 3, 3, 5\}]$(8g)

This also ends up with the following error message:

$$During evaluation of In[6] :=$$

$FindRoot::streg$: "$The starting point\ \{1.\}\ is not in the search region\ \{\{3.\}, \{5.\}\}. >>$.(8h)

3.1.7. Example 7:

$$In[7] =: FindRoot[Sin[\rho], \{\rho, 3.1, 3.2, 3, 5\}]$$

$Out[7] = \{\rho -> 3.14159\}$ (8i)

However, this ends up with correctly calculated result.

3.2.Finding Multiple roots by the Secant Method

Secant method whose Mathematica code is given above in (7) finds only one root in the interval: $[\rho_1, \rho_2]$. So, for finding multi-roots in the domain $[\rho_{min}, \rho_{max}]$ (which corresponds to $[0, R]$ here in our waveguide analyses), we can apply it by subdividing the domain into many sub-domains as follows:

Step 1:Define the followings:

Domain:$[\boldsymbol{\rho}_{\mathbf{min}}, \boldsymbol{\rho}_{\mathbf{max}}]$;number of sub-intervals to scan: $\boldsymbol{k}$.

Step 2: For $\boldsymbol{n} = \mathbf{1}$ to $\boldsymbol{k}$

$$\Delta\boldsymbol{\rho} = (\boldsymbol{\rho}_{\mathbf{max}} - \boldsymbol{\rho}_{\mathbf{min}})/\boldsymbol{k} \quad (9)$$

$\boldsymbol{\rho}_{\boldsymbol{i}}$ ←Find root by the Secant Method $\{\boldsymbol{f}(\boldsymbol{\rho}), \boldsymbol{\rho}_{\mathbf{1}\boldsymbol{i}} \leftarrow \boldsymbol{i} \times \Delta\boldsymbol{\rho}, \boldsymbol{\rho}_{\mathbf{2}\boldsymbol{i}} \leftarrow (\boldsymbol{i}+\mathbf{1}) \times \Delta\boldsymbol{\rho},$
$\boldsymbol{AccuracyGoal} \rightarrow \boldsymbol{prec}\mathbf{1}, \boldsymbol{WorkingPrecision} \rightarrow \boldsymbol{prec}\mathbf{2}\}$

End For

However, we have repeated roots some of which are very close to each other. Results of two examples where $\Delta\boldsymbol{\rho}$ is directly chosen as 0.2 to find the roots of $Sin(\rho)$ in $1 < \rho < 7$are given in Fig.3 and Fig.4.

```
In[8]:= Table[FindRoot[Sin[ρ], {ρ, n, n + 1}, Method → "Secant", AccuracyGoal → 30,
          WorkingPrecision → 30], {n, 1, 7, 0.2}]

Out[8]= {{ρ → -6.63024688346983829039579410624 × 10^-52},
         {ρ → 31.4159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338326},
         {ρ → 8.67352314982522083699226578305 × 10^-54}, {ρ → -417.831822927442500715531569976},
         {ρ → 12.5663706143591729538505735331}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656}}
```

Fig.3. Results for finding roots of $Sin(\rho)$ in $1 < \rho < 7$ using algorithm in (13) with $\Delta\boldsymbol{\rho} = 0.2$.

In the calculation shown in Fig.3, optional root interval is not defined in each step and we have two different but very close root values at $\rho \approx 0$. The same case also happens for the root $\rho \approx 3.14$. Moreover, we have roots far from the domain $1 \leq \rho \leq 7$. In the calculation shown in Fig.4, optional root interval is correctly defined as discussed above in each step. However, we again repeated roots and an error message for one of the roots as follows:

FindRoot::reged: "The point {3.2`_ is at the edge of the search region {3.2`30., 6.2`30._ in coordinate 1 and the computed search direction points outside the region.>>.

```
In[9]:= Table[FindRoot[Sin[ρ], {ρ, n, n + 1, n - 1, n + 2}, Method → "Secant", AccuracyGoal → 30,
          WorkingPrecision → 30], {n, 1, 7, 0.2}]

Out[9]= {{ρ → 0}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
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         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328},
         {ρ → 3.14159265358979323846264338328}, {ρ → 3.14159265358979323846264338328}, {ρ → 3.2},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656},
         {ρ → 6.28318530717958647692528676656}, {ρ → 6.28318530717958647692528676656}}
```

Fig.4. Results for finding roots of $Sin(\rho)$ in $1 < \rho < 7$ using algorithm in (13) with $\Delta\boldsymbol{\rho} = 0.2$ where root interval is defined in each step.

3.3. Our Secant Based Algorithm for the Circular Waveguides

We have used the sine function rather than Bessel functions of the first order (and also their derivatives) for simplification so far.We have seen that the secant method given above finds only one root in the interval: $[\boldsymbol{\rho_1}, \boldsymbol{\rho_2}]$ and the algorithm in (9) based on division of the domain into many small subdomains for finding multi-roots have some problems as discussed above. Convergence should also be considered carefully. Now, we suggest the following algorithm for the circular waveguides:

Step 1: Input the followings in manipulation console:
Guide radius: $\boldsymbol{R}$; frequency: $\boldsymbol{freq}$; medium permittivity and permeability values: $\boldsymbol{\varepsilon}$&$\boldsymbol{\mu}$;
Mode:$\boldsymbol{TE}$&$\boldsymbol{TM}$ (in two options); Bessel indice: $\boldsymbol{m}$; precision: $\boldsymbol{prec1}$;
number of sub-intervals to scan: $\boldsymbol{k}$.;number of digits: $\boldsymbol{prec}$2; update: $\boldsymbol{on/offbutton}$;
Matrix: $\boldsymbol{P_{k\times 1}} := (\boldsymbol{\rho_1 \rho_1} \dots \boldsymbol{\rho_k})^{\boldsymbol{T}}$
Step 2:*Evaluate and define the followings:*
If Mode=TM
$f(\rho) = J_m'(\rho)$
else
$f(\rho) = J_m(\rho)$
end if
domain of guide radius (ρ): $[0, R]$**;** $\Delta\boldsymbol{\rho} = \boldsymbol{R/k}$
Step 3: For $\boldsymbol{i = 1}$ to $\boldsymbol{k}$
$\boldsymbol{\rho_i}$ ←Find root by the SecantMethod {$\boldsymbol{f(\rho)}, \boldsymbol{\rho_{1i}} \leftarrow \boldsymbol{i} \times \Delta\boldsymbol{\rho}, \boldsymbol{\rho_{2i}} \leftarrow (\boldsymbol{i+1}) \times \Delta\boldsymbol{\rho},\ \boldsymbol{\rho_{min,i}} \leftarrow \boldsymbol{0}, \boldsymbol{\rho_{max,i}} \leftarrow \boldsymbol{R}, \boldsymbol{AccuracyGoal} \rightarrow \boldsymbol{prec1}, \boldsymbol{WorkingPrecision} \rightarrow \boldsymbol{prec2}$}
End For
Step 4: For $\boldsymbol{i = 1}$ to $\boldsymbol{k-1}$ do
For $\boldsymbol{j = i+1}$ to $\boldsymbol{k}$ do
If[Abs[Round[$\boldsymbol{\rho_i}$],0.001]-Round[$\boldsymbol{\rho_j}$],0.001]]<0.0001,
$\boldsymbol{\rho_i}$ ← “will be deleted”
End if
End for
End for
Step 5:$\boldsymbol{P} \leftarrow \boldsymbol{P} \setminus$ "$\boldsymbol{willbedeleted}$"
For $\boldsymbol{i = 1}$ to $\boldsymbol{n}$ ←Dimension of **P**
Evaluate frequencies: $\boldsymbol{f_c}$&$\boldsymbol{f_z}$ from mode $\boldsymbol{\rho_i} \in \boldsymbol{P}$**;** $\boldsymbol{P_i} \leftarrow (\boldsymbol{\rho_i f_c f_z})_{(\boldsymbol{1\times 3})}$
End for
Step 6: Output: Matrix **P** (10)

Accuracy goal value and working precision values are chosen as precision and output digit variables in the manipulation command of Mathematica, which is in the form as follows [21]:

$$\boldsymbol{Manipulate}[\boldsymbol{expr}, \{\{\boldsymbol{u}, \boldsymbol{u_{init}}\}, \boldsymbol{umin}, \boldsymbol{umax}, \boldsymbol{du}\}] \quad (11)$$

where variable $\boldsymbol{u}$ takes values between $\boldsymbol{umin}$ and $\boldsymbol{umax}$ with initial value:$\boldsymbol{u_{init}}$ and step:$\boldsymbol{du}$in the manipulation console. Our waveguide parameters and root finding parameters, are entered by the user via manipulation console in step 1 of our algorithm. Here, $\boldsymbol{i}$th element of column matrix $\boldsymbol{P}$is defined for $\boldsymbol{\rho_i}$ which is the $\boldsymbol{i}$th root of $\boldsymbol{f(\rho)}$ where index $\boldsymbol{i}$takes the value from zero up to the division number entered by the user as shown in Step 3. In step 2, function $\boldsymbol{f(\rho)}$ is defined: it is $\boldsymbol{J_m'(\rho)}$if the user selected mode as $\boldsymbol{TE}$ and it is $\boldsymbol{J_m(\rho)}$otherwise (which means mode is selected by the user as $\boldsymbol{TM}$ since it has defined in two options). Here, Bessel index $\boldsymbol{m}$ is also selected by the user in the manipulation console. In Step 3, domain $[\boldsymbol{0}, \boldsymbol{R}]$ is divided into user selected $\boldsymbol{k}$ sub-domains with index $\boldsymbol{i}$similar to (9). Here, Accuracy goal and working precision values are also selected by the user from the manipulation console. Result with $\boldsymbol{i}$th root is assigned to the $\boldsymbol{i}$th element of the column matrix $\boldsymbol{P}$. Here roots are found between $\boldsymbol{0 < \rho < R}$ in order that all the indices guarantee that initial point pairs are inside this domain to prevent an error. Roots outside the domain is automatically rejected. In Step 4, we scan all the elements except for the last one by index $\boldsymbol{i}$ and for each $\boldsymbol{i}$, we scan the other elements starting from the next element with index number $\boldsymbol{j}$ where $\boldsymbol{j}$ starts from $\boldsymbol{j = i+1}$and scans up to $\boldsymbol{k}$. Since the repeated roots in the result of the secant method are the same or very close to each other as seen in Fig. 3 and Fig. 4, we round the elements roots with index $\boldsymbol{i}$ and $\boldsymbol{j}$ up to 0.0001. If the absolute value of their differences is smaller than 0.001, either of them (it is the one with index $\boldsymbol{i}$ in our algorithm is replaced by the string element: “will be deleted” and the other one remains the same. In effect, repeated roots and roots very close to each other is eliminated. The numbers for rounding of and closeness decision given above are our preferences which works very well in the domain of the waveguide parameters selected by the user. For different domains, different preferences can be chosen. In Step 5, complement of the reduced column matrix (as a result of the eliminations of the repeated roots) is assigned to matrix $\boldsymbol{P}$, hence the elements with "will be deleted" are removed entirely. Now, we again scan all the elements in the reduced new $\boldsymbol{P}$ matrix from index $\boldsymbol{i = 1}$ up to $\boldsymbol{i = n}$ ←Dimension of **P**. Elements of $\boldsymbol{P}$(=new $\boldsymbol{\rho_i}$ elements) are now the desired distinct roots of $\boldsymbol{f(\rho)}$, which is whether $\boldsymbol{J'_m(\rho)}$or$\boldsymbol{J_m(\rho)}$, depending on the mode selection of the user. These roots are the values of the related modes ($\boldsymbol{\chi'_{mi} = TM_{mi}}$or $\boldsymbol{\chi_{mi} = TE_{mi}}$) where $\boldsymbol{m}$ value is the Bessel index selected by the user and $\boldsymbol{i}$ value is the position number of the mode in $\boldsymbol{P_{n\times1}}$. Now, related frequencies ($\boldsymbol{f_c}$&$\boldsymbol{f_z}$) are calculated according to (3c)&(3b) and (6c)-(6b) and $\boldsymbol{i}$th element of column matrix **P** is replaced by the triples: $\{\boldsymbol{\rho_i, f_c, f_z}\}$, which is a row matrix. As a result, matrix **P** becomes as follows:

$$\boldsymbol{P_{n\times3}} = \begin{pmatrix} \boldsymbol{\rho_1} & \boldsymbol{f_{c1}} & \boldsymbol{f_{z1}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\rho_n} & \boldsymbol{f_{cn}} & \boldsymbol{f_{zn}} \end{pmatrix} \tag{12}$$

For the user selected mode type ($\boldsymbol{TM}$ and $\boldsymbol{TE}$) and Bessel index $\boldsymbol{m}$, the solution of the waveguide under study becomes matrix P given in (12), namely,

$$\begin{aligned} &\boldsymbol{\chi'_{mi} = TM_{mi}}\text{or } \boldsymbol{\chi_{mi} = TE_{mi} \Rightarrow P_{i1}}, \quad \boldsymbol{i = 1, 2, \ldots, n} \\ &\boldsymbol{(f_c^{TM})_{mi}}\text{or}\boldsymbol{(f_c^{TE})_{mi} \Rightarrow P_{i2}}, \quad \boldsymbol{i = 1, 2, \ldots, n} \\ &\boldsymbol{(f_z^{TM})_{mi}}\text{or}\boldsymbol{(f_z^{TE})_{mi} \Rightarrow P_{i3}}, \quad \boldsymbol{i = 1, 2, \ldots, n} \end{aligned} \tag{13}$$

Guide wavelength given in (3d)-(6d) may also be added as the fourth column or any row in $\boldsymbol{P}$ in (12) may have not been added according to our aim to calculate.

IV. Results And Discussion

Our secant based design running under free Wolfram CDF player is available in [17] and entirely free for the researchers. Since the values of TM modes (χ_{mn}) and TE modes (χ'_{mn}) are the nth zeros of the Bessel functions and their derivatives with index m respectively, these values are conventional constants and independent of the waveguide parameters. One can check them by using our free design that these common values are in consistence with the results given in [6]. Although these common conventional values are available in such materials, their accuracy and fast calculation and speed of retrieving plays an important role here. In our design, user can change the accuracy by setting the precision value among: 10^{-12}, 10^{-11}, …,10^{-6} and can change the number of digits (=working precision) in the outputs among: 5,6,…,15. As the precision value is set to smaller values and/or the working precision is set to higher values, number of iterations in the secant based root finding whose Mathematica command is given in (7) increases and calculation speed decreases, though, we get much more precise results in much more digits. As any of the guide parameters (Guide radius: R, frequency: $freq$, medium permittivity and permeability values: ε&μ) change, related cut-off frequency: f_c and propagating wave frequency (in the z direction): f_z are instantly calculated for the desired TE or TM modes in the desired accuracy and in the desired digit numbers just after the update button is activated. Results for TE modes with some selected set values are given in Table 1, and results forTM modes with the same selected set values are given Table 2.

The TE and TM modes supported by the cylindrical waveguides are obtained successfully by the algorithm presented in (10) here. Results are in a great consistence with the results given in [6] where the Newton-Raphson method was used. Moreover, degenerate $\boldsymbol{TE_{0n}}$ and $\boldsymbol{TM_{1n}}$ modes where cut-off frequencies are common as given in [4-6] is also apparently seen (also shown in bold) in our results given in Table 1 and Table 2 with $\boldsymbol{\chi'_{0n} = \chi_{1n}}$. Note that, since roots of the related Bessel functions or their derivatives greater than the radius value of the CWG (which is set to 30 for Table 1 and Table 2 values) are not accessible, they are marked by 30< in Table 1 and Table 2. However, these modes in high $\boldsymbol{n}$ values become accessible and hence can be obtained in our design by increasing the radius of the CWG from the user console.This design works for lossless media and in high guide radius values but our aim here is just to find the roots by the conventional secant method and show their application to circular waveguides. However, it can be very easily modified for practical applications suitable for smaller guide radius values. As an example, radius can be scanned as if being huge according to our algorithm, all the way just as we did here, to determine the zeros. But, finally equation (3c) for the TE or equation (6c) for the TM is applied using the real radius values, which may be relatively very small.

Table.1. Results for TE modes with the values set to the followings: R=30cm, f=4GHz, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, precision=10^{-6}, division:100, number of output digits=10.

n \ m	0			1		
	χ'	f_c	f_z	χ'	f_c	f_z
1	**3.83170597**	**0.609413058**	**3.953304406**	1.841183781	0.292830777	3.989266867
2	**7.01558667**	**1.11579285**	**3.841224585**	5.331442773	0.847938456	3.909092014
3	**10.17346814**	**1.618037598**	**3.658135363**	8.536316366	1.357657059	3.762547981
4	**13.32369194**	**2.119064434**	**3.392575117**	11.7060049	1.861779661	3.540307401
5	**16.47063005**	**2.619568699**	**3.022889319**	14.86358863	2.363977056	3.226703035
6	**19.61585849**	**3.119801049**	**2.503366017**	18.01552786	2.865276722	2.79109106
7	**22.76008438**	**3.619873948**	**1.701914393**	21.16436986	3.366083789	2.160897944
8	**25.90367209**	**4.119845348**	**0**	24.31132686	3.866591057	1.024438185
9	**29.04682854**	**4.619748158**	**0**	27.45705057	4.366902178	0
n \ m	**2**			**3**		
	χ'	f_c	f_z	χ'	f_c	f_z
1	3.054236928	0.485760619	3.970395021	4.201188941	0.668177417	3.943797528
2	6.706133194	1.066575872	3.855180399	8.015236597	1.274782011	3.791428599
3	9.969467823	1.585592401	3.672314902	11.34592431	1.804510701	3.569837689
4	13.17037085	2.094679507	3.407685103	14.58584829	2.319803887	3.258605518
5	16.34752232	2.599989049	3.039746197	17.78874787	2.829208534	2.827645499
6	19.51291278	3.103428065	2.523635125	20.97247694	3.335564211	2.207716331
7	22.67158177	3.605798065	1.731536981	24.14489743	3.840121316	1.119583974
8	25.82603714	4.10749791	0	27.31005793	4.343523757	0
9	28.97767277	4.608749291	0	30<	-	-
n \ m	**4**			**5**		
	χ'	f_c	f_z	χ'	f_c	f_z
1	5.317553126	0.845729379	3.909570541	6.415616376	1.020370671	3.86766644
2	9.282396285	1.476317219	3.717591622	10.51986087	1.673129575	3.633268146
3	12.68190844	2.016992082	3.454235507	13.98718863	2.224590159	3.324334313
4	15.96410704	2.539008828	3.090863014	17.31284249	2.75351824	2.901402644
5	19.1960288	3.053029303	2.58437847	20.57551452	3.272429387	2.300262138
6	22.40103227	3.562768563	1.818427939	23.80358148	3.785836775	1.291293891
7	25.58975968	4.0699192	0	27.0103079	4.295850061	0
8	28.76783622	4.575375869	0	30<	-	-
9	30<	-	-	30<	-	-

Table.2. Results for TM modes with the values set to the followings: R=30cm, f=4GHz, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, precision=10^{-6}, division:100, number of output digits=10

n \ m	0			1		
	χ	f_c	f_z	χ	f_c	f_z
1	2.404825558	0.382475093	3.981672111	**3.83170597**	**0.609413058**	**3.953304406**
2	5.52007811	0.877939932	3.902463514	**7.01558667**	**1.11579285**	**3.841224585**
3	8.653727913	1.37633076	3.755757399	**10.17346814**	**1.618037598**	**3.658135363**
4	11.79153444	1.875382692	3.5331204	**13.32369194**	**2.119064434**	**3.392575117**
5	14.93091771	2.374685398	3.218830418	**16.47063005**	**2.619568699**	**3.022889319**
6	18.07106397	2.874109453	2.781994761	**19.61585849**	**3.119801049**	**2.503366017**
7	21.21163663	3.373601325	2.149142642	**22.76008438**	**3.619873948**	**1.701914393**
8	24.35247153	3.873134905	0.99941283	**25.90367209**	**4.119845348**	**0**
9	27.49347913	4.372695952	0	**29.04682854**	**4.619748158**	**0**
n \ m	**2**			**3**		
	χ	f_c	f_z	χ	f_c	f_z
1	5.135622302	0.81679422	3.915717968	6.380161896	1.014731819	3.869149691
2	8.41724414	1.338719236	3.769327633	9.76102313	1.552440349	3.686452083
3	11.61984117	1.848075764	3.547480228	13.01520072	2.070000499	3.422732525
4	14.79595178	2.353219764	3.234556653	16.22346616	2.580258558	3.056512028
5	17.95981949	2.856416593	2.800157897	19.40941523	3.086967313	2.543743856
6	21.11699705	3.358549389	2.172589699	22.58272959	3.591666584	1.760662133
7	24.27011231	3.860036097	1.048866689	25.7481667	4.095113018	0
8	27.42057355	4.361100696	0	28.90835078	4.597723985	0
9	30<	-	-	30<	-	-
n \ m	**4**			**5**		
	χ	f_c	f_z	χ'	f_c	f_z
1	7.588342434	1.206886698	3.813584206	8.771483816	1.395059228	3.748841121
2	11.06470949	1.759784935	3.592096461	12.3386042	1.962391314	3.485544481
3	14.37253667	2.285877776	3.282493381	15.70017408	2.497031653	3.124873265
4	17.61596605	2.801728478	2.854876098	18.98013388	3.01869233	2.62440405
5	20.82693296	3.312416192	2.2422977	22.2177999	3.5336264	1.874429104
6	24.01901952	3.820101085	1.186097678	25.43034115	4.044564506	0
7	27.19908777	4.325874525	0	28.62661831	4.552915889	0
8	30<	-	-	30<	-	-
9	30<	-	-	30<	-	-

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